

Proof of Kolmogorovian Censorship

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Abstract

Many argued (Accardi and Fedullo, Pitowsky) that Kolmogorov's axioms of classical probability theory are incompatible with quantum probabilities, and this is the reason for the violation of Bell's inequalities. Szabó showed that, in fact, these inequalities are not violated by the experimentally observed frequencies if we consider the real, "effective" frequencies. We prove in this work a theorem which generalizes this result: "effective" frequencies associated to quantum events always admit a Kolmogorovian representation, when these events are collected through different experimental set ups, the choice of which obeys a classical distribution.

1 Introduction

It is commonly accepted that the frequencies observed during the so-called Orsay experiments (which agree with the quantum predictions) violate the Clauser-Horne inequalities. Pitowsky (1989) proved that the fulfillment of these inequalities would be a necessary and sufficient condition for the existence of a Kolmogorovian representation for these frequencies (probabilities). Szabó (1995a,b) has recently shown, that the *effective* frequencies observed

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in the Orsay experiments do no longer violate the Clauser-Horne inequalities. These effective frequencies are obtained by multiplying the quantum probabilities by the relative frequencies associated to the stochastic devices making choice among the polarizer orientations. In an endnote of the article (Szabó 1995b), the author formulated the so-called “Kolmogorovian Censorship” hypothesis:

“On the basis of particular examples, it seems that there is a ‘Kolmogorovian Censorship’ in the real world: We never encounter ‘naked’ quantum probabilities in reality. A correlation vector consisting of empirically testable probabilities is always a product

$$(p_1 \dots p_n \dots p_{ij} \dots) = (\pi_1 \dots \pi_n \dots \pi_{ij} \dots) \cdot (\tilde{p}_1 \dots \tilde{p}_n \dots \tilde{p}_{ij} \dots) \\ = (\pi_1 \tilde{p}_1 \dots \pi_n \tilde{p}_n \dots \pi_{ij} \tilde{p}_{ij} \dots),$$

where $(\pi_1 \dots \pi_n \dots \pi_{ij})$ are quantum probabilities and $(\tilde{p}_1 \dots \tilde{p}_n \dots \tilde{p}_{ij})$ are classical probabilities with which the corresponding measurements happen to be performed. My conjecture is that such a product is always classical. (From pure mathematical point of view, a product of a quantum and a classical correlation vector is not necessarily classical.)”

We shall first (section 3) describe the Orsay experiments and introduce the formalism of Pitowsky. We shall then show how to generalize the Pitowsky formalism for conjunctions of more than two events. In section 4 we prove the validity of Szabó’s “Kolmogorovian Censorship” formulating it as a clear mathematical statement.

2 The Orsay experiments, the Pitowsky polytope and the Clauser-Horne’s inequalities.

2.1 The Orsay experiments.

The experiments realised in Orsay by Aspect et al. in order to test Bell’s inequalities (and also Clauser-Horne inequalities which are a variant of Bell’s inequalities) proceed as follows. A source emits two photons along opposite directions. Two polarisers are placed in two spatially separated regions (Left and Right), symmetrically, on both sides of the source. A polarizer measures a dichotomic variable, the sign of the linear polarisation of

the incoming photon along a direction in the plane perpendicular to its direction of propagation. The experimenter chooses, for each polariser, one direction for the measurement of polarisation between two different possible directions : the directions a and a' in the Left region, b and b' in the Right region. The technical details are not important here, but it is worth to know that for some well chosen directions of the polarisers¹, we obtain by orthodox quantum mechanical computations that the ‘naked’ probabilities $(P(A), P(A'), P(B), P(B'), P(A \cap B), P(A \cap B'), P(A' \cap B), P(A' \cap B'))$ yield $(1/2, 1/2, 1/2, 1/2, \frac{\sin^2(\pi/8)}{2}, \frac{\sin^2(\pi/8)}{2}, \frac{\sin^2(3\pi/8)}{2}, \frac{\sin^2(\pi/8)}{2})$, where A (A' , B , B') represent the property “the photon has + polarisation along the a (a' , b , b') direction”. These probabilities were observed as experimental frequencies, with a very good precision.

In order to remain coherent with Szabó’s notations, we shall not consider this experiment, but a similar one where the polarisations are replaced by spins one half and the polarisers are replaced by Stern-Gerlach magnets. The initial state is the singlet state. There are four magnets altogether (two on both sides) and they detect the spin-up events. Two switches, one for each particle, are making choice from sending them to the Stern-Gerlach magnets directed into different directions with probabilities 0.5-0.5. The observed events are the following:

- A : The “left particle has spin ‘up’ into direction \mathbf{a} ” detector beeps
- A' : The “left particle has spin ‘up’ into direction \mathbf{a}' ” detector beeps
- B : The “right particle has spin ‘up’ into direction \mathbf{b} ” detector beeps
- B' : The “right particle has spin ‘up’ into direction \mathbf{b}' ” detector beeps
- a : The left switch selects direction \mathbf{a}
- a' : The left switch selects direction \mathbf{a}'
- b : The right switch selects direction \mathbf{b}
- b' : The right switch selects direction \mathbf{b}'

For the probabilities of these events, in case of $\theta(\mathbf{a}, \mathbf{a}') = \theta(\mathbf{a}', \mathbf{b}') = \theta(\mathbf{a}, \mathbf{b}') = 120^\circ$ and $\theta(\mathbf{b}, \mathbf{a}') = 0$, we have

$$\begin{aligned} p(A) = p(A') = p(B) = p(B') &= \frac{1}{4} \\ p(a) = p(a') = p(b) = p(b') &= \frac{1}{2} \end{aligned}$$

¹The source emits a pair of photons forming an entangled state describable by the singlet state, the directions a , a' , b , b' are coplanar and are all separated by angles of 22.5 degrees, in the order a' , b' , a , b .

$$\begin{aligned}
p(A \wedge a) = p(A) &= \frac{1}{4} \\
p(A' \wedge a') = p(A') &= \frac{1}{4} \\
p(B \wedge b) = p(B) &= \frac{1}{4} \\
p(B' \wedge b') = p(B') &= \frac{1}{4} \\
p(A \wedge a') = p(A' \wedge a) = p(B \wedge b') = p(B' \wedge b) &= 0 \\
p(A \wedge B) = p(A \wedge B') = p(A' \wedge B') &= \frac{3}{32} \\
p(A' \wedge B) &= 0 \\
p(a \wedge a) = p(b \wedge b') &= 0 \\
p(a \wedge b) = p(a \wedge b') = p(a' \wedge b) = p(a' \wedge b') &= \frac{1}{4} \\
p(A \wedge b) = p(A \wedge b') = p(A' \wedge b) = p(A' \wedge b') \\
= p(B \wedge a) = p(B \wedge a') = p(B' \wedge a) = p(B' \wedge a') &= \frac{1}{8}
\end{aligned} \tag{1}$$

These statistical data agree with quantum mechanical results, in the sense that

$$\begin{aligned}
\frac{p(A \wedge a)}{p(a)} &= \text{tr}(\hat{W} \hat{A}) = \frac{p(A' \wedge a')}{p(a')} = \text{tr}(\hat{W} \hat{A}') \\
= \frac{p(B \wedge b)}{p(b)} &= \text{tr}(\hat{W} \hat{B}) = \frac{p(B' \wedge b')}{p(b')} = \text{tr}(\hat{W} \hat{B}') = \frac{1}{2} \\
\frac{p(A \wedge B \wedge a \wedge b)}{p(a \wedge b)} &= \frac{p(A \wedge B)}{p(a \wedge b)} = \text{tr}(\hat{W} \hat{A} \hat{B}) \\
&= \frac{1}{2} \sin^2 \frac{1}{2} \theta(\mathbf{a}, \mathbf{b}) = \frac{3}{8} \\
\frac{p(A \wedge B' \wedge a \wedge b')}{p(a \wedge b')} &= \frac{p(A \wedge B')}{p(a \wedge b')} = \text{tr}(\hat{W} \hat{A} \hat{B}') \\
&= \frac{1}{2} \sin^2 \frac{1}{2} \theta(\mathbf{a}, \mathbf{b}') = \frac{3}{8} \\
\frac{p(A' \wedge B \wedge a' \wedge b)}{p(a' \wedge b)} &= \frac{p(A' \wedge B)}{p(a' \wedge b)} = \text{tr}(\hat{W} \hat{A}' \hat{B}) \\
&= \frac{1}{2} \sin^2 \frac{1}{2} \theta(\mathbf{a}', \mathbf{b}) = 0 \\
\frac{p(A' \wedge B' \wedge a' \wedge b')}{p(a' \wedge b')} &= \frac{p(A' \wedge B')}{p(a' \wedge b')} = \text{tr}(\hat{W} \hat{A}' \hat{B}')
\end{aligned} \tag{2}$$

$$= \frac{1}{2} \sin^2 \frac{1}{2} \theta(\mathbf{a}', \mathbf{b}') = \frac{3}{8}$$

where the outcomes are identified with the following projectors

$$\begin{aligned}\hat{A} &= \hat{P}_{\text{span}\{\psi_{+\mathbf{a}} \otimes \psi_{+\mathbf{a}}, \psi_{+\mathbf{a}} \otimes \psi_{-\mathbf{a}}\}} \\ \hat{A}' &= \hat{P}_{\text{span}\{\psi_{+\mathbf{a}'} \otimes \psi_{+\mathbf{a}'}, \psi_{+\mathbf{a}'} \otimes \psi_{-\mathbf{a}'}\}} \\ \hat{B} &= \hat{P}_{\text{span}\{\psi_{-\mathbf{b}} \otimes \psi_{+\mathbf{b}}, \psi_{+\mathbf{b}} \otimes \psi_{+\mathbf{b}}\}} \\ \hat{B}' &= \hat{P}_{\text{span}\{\psi_{-\mathbf{b}'} \otimes \psi_{+\mathbf{b}'}, \psi_{+\mathbf{b}'} \otimes \psi_{+\mathbf{b}'}\}}\end{aligned}$$

of the Hilbert space $H^2 \otimes H^2$, and where the singlet state is represented as $\hat{W} = \hat{P}_{\Psi_s}$, where $\Psi_s = \frac{1}{\sqrt{2}} (\psi_{+\mathbf{a}} \otimes \psi_{-\mathbf{a}} - \psi_{-\mathbf{a}} \otimes \psi_{+\mathbf{a}})$.

Remark : The probabilities $P(A' \wedge A)$ and $P(B \wedge B')$ are not taken into account because the choice of a direction for a Stern-Gerlach magnet excludes the other direction, we cannot measure a and a' (b and b') simultaneously, the corresponding operators do not commute. We will now recall some important results of Pitowsky.

2.2 The Pitowsky formalism

The question whether given probabilities are representable in a Kolmogorovian probability model or not can be completely answered. Pitowsky (1989) elaborated a convenient geometric language for the discussion of this problem and proved a theorem providing the necessary and sufficient condition of such a representation. In this section we recall the basic elements of his formalism and present the theorem. We also prove a straightforward generalization of Pitowsky's original theorem for the case of conjunctions of more than two events.

Let S be a set of pairs of integers $S \subset \{\{i, j\} \mid 1 \leq i < j \leq n\}$. Denote by $R(n, S)$ the linear space of real vectors having a form like $(f_1, f_2, \dots, f_n, \dots, f_{ij}, \dots)$, $\{i, j\} \in S$. For each $\varepsilon \in \{0, 1\}^n$, let u^ε be the following vector in $R(n, S)$:

$$\begin{aligned}u_i^\varepsilon &= \varepsilon_i, & 1 \leq i \leq n, \\ u_{ij}^\varepsilon &= \varepsilon_i \varepsilon_j, & \{i, j\} \in S.\end{aligned}$$

Definition 2.1 The classical correlation polytope $\mathcal{C}(n, S)$ is the closed convex hull of vectors $\{u^\varepsilon\}_{\varepsilon \in \{0,1\}^n}$ in $R(n, S)$:

$$\mathcal{C}(n, S) := \left\{ a \in R(n, S) \mid a = \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon u^\varepsilon, \text{ where } \lambda_\varepsilon \geq 0 \text{ and } \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon = 1 \right\}$$

Consider now events A_1, A_2, \dots, A_n and some of their conjunctions $A_i \wedge A_j$ ($\{i, j\} \in S$). Assume that we can associate probabilities to them (that is, we order numbers to them about which we think that they could be probabilities), from which we can form a so called correlation vector:

$$\begin{aligned} \mathbf{p} &= (p_1, p_2, \dots, p_n, \dots, p_{ij}, \dots) \\ &= (p(A_1), p(A_2), \dots, p(A_n), \dots, p(A_i \wedge A_j), \dots) \in R(n, S) \end{aligned}$$

We will then say that

Definition 2.2 \mathbf{p} has a Kolmogorovian representation if there exist a Kolmogorovian probability space (Ω, Σ, μ) and measurable subsets

$$X_{A_1}, X_{A_2}, \dots, X_{A_n} \in \Sigma$$

such that

$$\begin{aligned} p_i &= \mu(X_{A_i}), & 1 \leq i \leq n, \\ p_{ij} &= \mu(X_{A_i} \cap X_{A_j}), & \{i, j\} \in S. \end{aligned}$$

The following theorem due to Pitowsky (Pitowsky 1989) allows us to formulate the existence of a Kolmogorovian representation for a correlation vector in terms of a geometrical condition.

Theorem 2.1 A correlation vector $\mathbf{p} = (p_1, p_2, \dots, p_n, \dots, p_{ij}, \dots)$ has a Kolmogorovian representation if and only if this vector belongs to the classical polytope ($\mathbf{p} \in \mathcal{C}(n, S)$).

In case $n = 4$ and $S = S_4 = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$, the condition $\mathbf{p} \in \mathcal{C}(n, S)$ can be shown (Pitowsky 1989) to be equivalent to the following

inequalities:

$$\begin{aligned}
0 &\leq p_{ij} \leq p_i \leq 1, \\
0 &\leq p_{ij} \leq p_j \leq 1, \quad i = 1, 2 \quad j = 3, 4 \\
p_i + p_j - p_{ij} &\leq 1, \\
-1 &\leq p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leq 0, \\
-1 &\leq p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 \leq 0, \\
-1 &\leq p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 \leq 0, \\
-1 &\leq p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 \leq 0.
\end{aligned} \tag{3}$$

These last four equations are equivalent to the well known Clauser-Horne inequalities (Clauser, Horne 1974), which are a variant of Bell's inequalities (Bell 1964).

2.3 The generalized Pitowsky theorem

Pitowsky's original theorem deals with simple conjunctions only. We present here a straightforward generalization of the theorem for the case where conjunctions of not only two but three or more events are considered.

A typical correlation vector is then:

$$(p_1, \dots, p_n, \dots, p_{i_1 i_2}, \dots, p_{j_1 j_2 j_3}, \dots, p_{k_1 k_2 k_3 k_4} \dots) \tag{4}$$

where $p_{j_1 j_2 j_3}$, $p_{k_1 k_2 k_3 k_4}$ etc. stand for probabilities of conjunctions of three or more events. More precisely, consider a set S of subsets of indexes $S \subset 2^{\{1, \dots, n\}} \setminus \{\emptyset\}$, where we denote by $2^{\{1, \dots, n\}}$ the power set of $\{1, \dots, n\}$.

Remark: In the formalism of Pitowsky, the set (S) of indices related to the conjunctions does not contain the set of n pure indices (i from 1 to n). In the notation introduced here, S contains also the set of isolated indices i (i from 1 to n) related to one property only. This way the notations become simpler and one does not necessarily have to have assumptions for the probabilities of A_i -s.

Consider the linear space of real functions

$$\mathbf{R}(n, S) = \{\mathbf{f} \mid S \ni I \mapsto \mathbf{f}_I \in \mathbf{R}\}$$

For each $\varepsilon \in \{0, 1\}^n$ we define $\mathbf{u}^\varepsilon \in \mathbf{R}(n, S)$ as

$$\mathbf{u}_I^\varepsilon = \prod_{i \in I} \varepsilon_i, \quad (\forall I \in S)$$

Definition 2.3 The generalized classical correlation polytope $\mathbf{C}(n, S)$ is the closed convex hull of vectors $\{\mathbf{u}^\varepsilon\}_{\varepsilon \in \{0,1\}^n}$ in $\mathbf{R}(n, S)$:

$$\mathbf{C}(n, S) := \left\{ \mathbf{a} \in \mathbf{R}(n, S) \mid \mathbf{a} = \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon \mathbf{u}^\varepsilon, \text{ where } \lambda_\varepsilon \geq 0 \text{ and } \sum_{\varepsilon \in \{0,1\}^n} \lambda_\varepsilon = 1 \right\}$$

Consider now conjunctions $\bigwedge_{i \in I} A_i$ ($I \in S$) of events A_1, A_2, \dots, A_n . If we associate probabilities to them (that is numbers about which we think that they can be the probabilities of the events but they don't have to come from some well defined probability theory) we can form a generalized correlation vector $(\mathbf{p}, \{A_i\}_{i=1}^n)_S$ where $\mathbf{p} \in \mathbf{R}(n, S)$:

$$\mathbf{p}_I = p \left(\bigwedge_{i \in I} A_i \right) \quad (5)$$

In the following we will sometimes refer to $(\mathbf{p}, \{A_i\}_{i=1}^n)_S$ only by \mathbf{p} .

Definition 2.4 A generalized correlation vector $(\mathbf{p}, \{A_i\}_{i=1}^n)_S$ has a Kolmogorovian representation if there exist a Kolmogorovian probability space (Ω, Σ, μ) and measurable subsets

$$X_{A_1}, X_{A_2}, \dots, X_{A_n} \in \Sigma$$

such that

$$\mathbf{p}_I = \mu \left(\bigcap_{i \in I} X_{A_i} \right), \quad (\forall I \in S).$$

We denote the representation with $(\Omega, \Sigma, \mu, \{X_{A_i}\}_{i=1}^n)$

Now we can formulate the generalization of Pitowsky's theorem and prove it by a straightforward generalisation of the original proof (Pitowsky 1989).

Theorem 2.2 A generalized correlation vector $\mathbf{p} \in \mathbf{R}(n, S)$ has a Kolmogorovian representation if and only if it belongs to the generalised classical polytope $(\mathbf{p} \in \mathbf{C}(n, S))$.

Proof Assume, \mathbf{p} has a Kolmogorovian representation $(\Omega, \Sigma, \mu \{X_{A_i}\}_{i=1}^n)$. For any ε in $\{0, 1\}^n$ we define a measurable subset $X^\varepsilon = \bigcap_{0 \leq i \leq n} X_{A_i}^{\varepsilon_i}$, where $X_{A_i}^1 = X_{A_i}$; $X_{A_i}^0 = \Omega \setminus X_{A_i}$. One can easily check that if $\varepsilon^1, \varepsilon^2 \in \{0, 1\}^n$, $\varepsilon^1 \neq \varepsilon^2$ then $X^{\varepsilon^1} \cap X^{\varepsilon^2} = \emptyset$, $\bigcup_{\varepsilon \in \{0, 1\}^n} X^\varepsilon = \Omega$. Moreover, for all $I \in S$,

$$\bigcap_{i \in I} X_{A_i} = \bigcup_{\substack{\varepsilon \in \{0, 1\}^n \\ \varepsilon_i = 1 \text{ if } i \in I}} X^\varepsilon$$

Therefore

$$\begin{aligned} \mathbf{p}_I &= \mu\left(\bigcap_{i \in I} X_{A_i}\right) = \sum_{\substack{\varepsilon \in \{0, 1\}^n \\ \varepsilon_i = 1 \text{ if } i \in I}} \mu(X^\varepsilon) \\ &= \sum_{\varepsilon \in \{0, 1\}^n} \mu(X^\varepsilon) \cdot \prod_{i \in I} \varepsilon_i = \sum_{\varepsilon \in \{0, 1\}^n} \mu(X^\varepsilon) \cdot \mathbf{u}_I^\varepsilon. \end{aligned}$$

This means that \mathbf{p} is a convex linear combination of the vertices \mathbf{u}^ε with weights $\lambda_\varepsilon = \mu(X^\varepsilon)$.

Assume now that \mathbf{p} is a convex linear combination of the vertices,

$$\mathbf{p} = \sum_{\varepsilon \in \{0, 1\}^n} \lambda_\varepsilon \cdot \mathbf{u}^\varepsilon.$$

Let $\Omega = \{0, 1\}^n$. The Kolmogorovian representation can be based on subsets $X_{A_i} = \{\varepsilon \in \{0, 1\}^n \mid \varepsilon_i = 1\}$. Then, for every $I \in S : \bigcap_{i \in I} X_{A_i} = \{\varepsilon \in \{0, 1\}^n \mid \prod_{i \in I} \varepsilon_i = 1\}$. Then, with the previous notation, $X^\varepsilon = \{\varepsilon\}$. Let Σ be the power set of Ω . The measure of an arbitrary $X \in \Sigma$ is defined as $\mu(X) = \sum_{\varepsilon \in X} \lambda_\varepsilon$. It is easy to check that this is a probability measure. We have then

$$\mu\left(\bigcap_{i \in I} X_{A_i}\right) = \sum_{\varepsilon \in \{0, 1\}^n} \mu(\{\varepsilon\}) \cdot \mathbf{u}_I^\varepsilon = \mathbf{p}_I,$$

which proves that the correlation vector admits well a Kolmogorovian representation.

□

Let us now apply the Pitowsky-Clauser-Horne inequalities (3) to check if a Kolmogorovian representation exists for the ‘naked’ quantum probabilities

associated to the measurement of spin directions introduced in subsection “The Orsay experiments”.

$$p_1 = tr(\hat{W}\hat{A}), p_2 = tr(\hat{W}\hat{A}'), p_3 = tr(\hat{W}\hat{B}), p_4 = tr(\hat{W}\hat{B}'),$$

$$p_{13} = tr(\hat{W}\hat{A}\hat{B}), p_{14} = tr(\hat{W}\hat{A}\hat{B}'), p_{23} = tr(\hat{W}\hat{A}'\hat{B}), p_{24} = tr(\hat{W}\hat{A}'\hat{B}')$$

Substituting the values obtained in (2) for the probabilities in the last inequality of (3) we find

$$\frac{3}{8} + \frac{3}{8} + \frac{3}{8} - 0 - \frac{1}{2} - \frac{1}{2} = \frac{1}{8} > 0.$$

Consequently,

$$\mathbf{p} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, 0, \frac{3}{8} \right) \notin \mathcal{C}(n, S). \quad (6)$$

The *usual* conclusion is that the observed probabilities in the Orsay experiment have no Kolmogorovian representation. However, as Szabó (Szabó 1995a,b) pointed out, a closer analysis can yield different conclusion.

The problem, he claims, is that probabilities in (6) are not the effective relative frequencies of the observed events, the values of which are given in (1), but the conditional probabilities (2). The meaning of such a conditional probability is this: the probability of a measurement outcome, given that the corresponding measurement has been completed. He proved that the effective probabilities we encounter in the Orsay experiment, that is the values in (1), can be accommodated into a Kolmogorovian theory. For instance, he showed that the effective relative frequencies given in (1) do no longer violate the Clauser-Horne inequalities, as shows the following:

$$-1 < \frac{3}{32} + \frac{3}{32} + \frac{3}{32} - 0 - \frac{1}{4} - \frac{1}{4} = \frac{-7}{32} < 0.$$

He also showed by numerical methods (Szabó 1995a,b) that the correlation vector presented in (1), which contains the effective probabilities of the events $a, a', b, b', A, A', B, B'$ admits a Kolmogorovian representation. A generalisation of Szabó’s case (which is valid for a particular choice for the directions a, a', b, b') to a situation which covers all possible choices of such directions is given in (Durt 1995 a), but suffers from a lack of generality (it is limited to an Orsay like situation). We shall now generalize these results, in conformity with the Kolmogorovian censorship hypothesis mentioned in the introduction.

3 Kolmogorovian Censorship

3.1 Introduction, a fundamental remark.

Let us represent by a density operator \hat{W} the state of the quantum system under measurement. n different measurements are carried out on it. A measurement inspects whether the value of an observable is in a set or not. The outcome can be yes or no. Let a_i denote the event “the i -th measurement has been performed”. The event when the result of measurement a_i is yes is denoted by A_i . The corresponding projectors are \hat{A}_i , which are given by the spectral decompositions of the observables. We can now formulate a remark which will appear to be essential in the proof of the main theorem:

Fundamental remark: Among the measurements there may be incompatible ones, but those that are carried out simultaneously must be, and actually are, compatible.

In our terminology compatibility of measurements means that the corresponding operators commute. According to orthodox quantum physics only these measurements can be performed together (the projectors which are associated to sets of outcomes of the same observable necessarily commute because the basis associated to the spectral decomposition is orthocomplemented.)

The structure of the proof is the following.

We shall express in the forthcoming subsection (4.2) the constraints imposed by the fact that the probability distribution of the performances of the measurements is classical and that during simultaneous measurements, the quantum properties are represented by compatible (commuting) projectors. Afterwards, we shall recall that the naked quantum probabilities associated to compatible projectors admit a Kolmogorovian representation (4.3). This will allow us to give a compact expression of the effective probabilities (4.4), and to prove the main theorem (4.5).

3.2 Constraints on the probability distribution of the performances of the measurements

First assumption:

The general correlation vector $\tilde{\mathbf{p}}$ consisting of the probabilities with which the measurements are performed, that is the one with components $p(\bigwedge_{i \in I} a_i)$ where $I \in 2^{\{1, \dots, n\}} \setminus \{\emptyset\}$ is supposed to be classical. This assumption is very natural, because after all the devices used to choose which measurement is performed in the laboratory are macroscopical devices of

classically describable nature (it is also so in the Orsay experiments), consequently

$$\tilde{\mathbf{p}} = \sum_{\varepsilon \in \{0,1\}^n} \kappa_\varepsilon \mathbf{u}^\varepsilon, \quad \sum_{\varepsilon \in \{0,1\}^n} \kappa_\varepsilon = 1, \quad \kappa_\varepsilon \geq 0 \quad (7)$$

Second assumption: In conformity with the fundamental remark, we assume that from incompatibility of two measurements i, j (i.e. $[\hat{A}_i, \hat{A}_j] \neq 0$) follows that they are not performed together.

This implies that a restriction on $\tilde{\mathbf{p}}$ occurs. Let us introduce the set of \mathcal{K} of indices which represent compatible measurements:

$$\mathcal{K} = \left\{ I \in 2^{\{1, \dots, n\}} \setminus \{\emptyset\} \mid \forall i, j \in I : [\hat{A}_i, \hat{A}_j] = 0 \right\}.$$

Let $\varepsilon^I \in \{0, 1\}^n$ mean the following: $\forall I \in 2^{\{1, \dots, n\}} \setminus \{\emptyset\} : \varepsilon_i^I = 1 \Leftrightarrow i \in I$. This is a one to one correspondence between $2^{\{1, \dots, n\}} \setminus \{\emptyset\}$ and $\{0, 1\}^n$.

The second assumption can now be formulated as follows: $I \notin \mathcal{K} \Rightarrow \tilde{\mathbf{p}}_I = 0$. From these assumptions follows that we can restrict the expression of the vector $\tilde{\mathbf{p}}$ as we show it now.

Proposition 3.1 *If we assume (assumption 1) that $\tilde{\mathbf{p}}$, the probability distribution of the performances of the measurements is classical and has the decomposition (7), if, furthermore, we assume (assumption 2) that from incompatibility of two measurements i, j (i.e. $[\hat{A}_i, \hat{A}_j] \neq 0$) follows that they are not performed together (i.e. $I \notin \mathcal{K} \Rightarrow \tilde{\mathbf{p}}_I = 0$) then*

$$\tilde{\mathbf{p}} = \sum_{I \in \mathcal{K}} \kappa_{\varepsilon^I} \mathbf{u}^{\varepsilon^I}.$$

Proof: It is generally true and follows from decomposition (7) that $\tilde{\mathbf{p}}_I \geq \kappa_{\varepsilon^I} \mathbf{u}_I^{\varepsilon^I} = \kappa_{\varepsilon^I}$. But, if $I \notin \mathcal{K}$ then $\tilde{\mathbf{p}}_I = 0$ and so $\kappa_{\varepsilon^I} = 0$. This means exactly that

$$\tilde{\mathbf{p}} = \sum_{\varepsilon \in \{0,1\}^n} \kappa_\varepsilon \mathbf{u}^\varepsilon = \sum_{I \in \mathcal{K}} \kappa_{\varepsilon^I} \mathbf{u}^{\varepsilon^I}.$$

Consequently

$$p\left(\bigwedge_{j \in I} a_j\right) = \sum_{J \in \mathcal{K} : I \subset J} \kappa_{\varepsilon^J}.$$

3.3 A Kolmogorovian representation for naked quantum probabilities of compatible projectors.

We will reproduce the proof of the well-known fact that the naked quantum probabilities associated to commuting projectors is Kolmogorovian by showing a simple explicit representation.

Proposition 3.2 *For all $J \in \mathcal{K}$ the correlation vector*

$$\pi^J : 2^J \setminus \{\emptyset\} \ni I \rightarrow \pi_I^J = \text{tr} \left(\hat{W} \prod_{i \in I} \hat{A}_i \right)$$

is Kolmogorovian.

A possible representation is the following:

$$\begin{aligned} \Omega^J &= \{0, 1\}^{\text{card} J} \\ \Sigma^J &= 2^{\Omega^J} \\ \forall \epsilon \in \Omega^J : \mu^J(\{\epsilon\}) &= \text{tr} \left(\hat{W} \prod_{i \in J} \hat{A}_i^{\epsilon_i} \right) \end{aligned}$$

where \hat{A}_i^0 means the orthogonal complement of \hat{A}_i .

$$\begin{aligned} X_{A_i}^J &= \left\{ \epsilon \in \Omega^J \mid \epsilon_i = 1 \right\}, \quad i \in J \\ \pi_I^J &= \mu \left(\bigcap_{i \in I} X_{A_i}^J \right) \end{aligned}$$

It can be proven (Durt 1996a), by making use of the compatibility of the projectors involved in the representation and of the properties of the density matrix that μ^J satisfies the definition of a Kolmogorovian measure.

As an example consider the Orsay experiment. We notice that the measurements “ a ” as well as “ a' ” can be performed simultaneously with “ b ” or “ b' ”, the corresponding operators commute. In this case, the correlation vectors

$$\begin{aligned} \mathbf{p}^{ab} &= \left(\text{tr}(\hat{W} \hat{A}), \text{tr}(\hat{W} \hat{B}), \text{tr}(\hat{W} \hat{A} \hat{B}) \right) \\ \mathbf{p}^{ab'} &= \left(\text{tr}(\hat{W} \hat{A}), \text{tr}(\hat{W} \hat{B}'), \text{tr}(\hat{W} \hat{A} \hat{B}') \right) \\ \mathbf{p}^{a'b} &= \left(\text{tr}(\hat{W} \hat{A}'), \text{tr}(\hat{W} \hat{B}), \text{tr}(\hat{W} \hat{A}' \hat{B}) \right) \\ \mathbf{p}^{a'b'} &= \left(\text{tr}(\hat{W} \hat{A}'), \text{tr}(\hat{W} \hat{B}'), \text{tr}(\hat{W} \hat{A}' \hat{B}') \right) \end{aligned} \tag{8}$$

have Kolmogorovian representations:

$$\begin{aligned}
& \left(\Omega^{ab}, \Sigma^{ab}, \mu^{ab}, \{X_A^{ab}, X_B^{ab}\} \right) \\
& \left(\Omega^{ab'}, \Sigma^{ab'}, \mu^{ab'}, \{X_A^{ab'}, X_{B'}^{ab'}\} \right) \\
& \left(\Omega^{a'b}, \Sigma^{a'b}, \mu^{a'b}, \{X_{A'}^{a'b}, X_B^{a'b}\} \right) \\
& \left(\Omega^{a'b'}, \Sigma^{a'b'}, \mu^{a'b'}, \{X_{A'}^{a'b'}, X_{B'}^{a'b'}\} \right)
\end{aligned} \tag{9}$$

respectively. These representations are shown in Table 1, corresponding to the particular choice of directions a, a', b, b' , made at the beginning of this work.

$a \cap b$		$a \cap b'$	
$A \cap B$ $\frac{3}{8}$	$A \cap \neg B$ $\frac{1}{8}$	$A \cap B'$ $\frac{3}{8}$	$A \cap \neg B'$ $\frac{1}{8}$
$\neg A \cap B$ $\frac{1}{8}$	$\neg A \cap \neg B$ $\frac{3}{8}$	$\neg A \cap B'$ $\frac{1}{8}$	$\neg A \cap \neg B'$ $\frac{3}{8}$
$a' \cap b$		$a' \cap b'$	
$A' \cap B$ 0	$A' \cap \neg B$ $\frac{1}{2}$	$A' \cap B'$ $\frac{3}{8}$	$A' \cap \neg B'$ $\frac{1}{8}$
$\neg A' \cap B$ $\frac{1}{2}$	$\neg A' \cap \neg B$ 0	$\neg A' \cap B'$ $\frac{1}{8}$	$\neg A' \cap \neg B'$ $\frac{3}{8}$

Table 1: The Kolmogorovian representations of the “naked” Orsay frequencies for the four choices of experimental arrangements.

3.4 A compact expression of the effective probabilities.

As we already emphasised, the “naked” quantum probabilities $tr(\hat{W}\hat{A}_1)$, $tr(\hat{W}\hat{A}_2)$, ..., $tr(\hat{W}\hat{A}_i\hat{A}_j)$ are conditional probabilities. In order to get the probabilities of the outcomes we must multiply these values by the probabilities of the performance of the corresponding measurements. That is, the

effective probability of, for instance A_1 , is

$$p(A_1) = p(A_1 \wedge a_1) = p(A_1 \mid a_1) \cdot p(a_1) = p(a_1) \cdot \text{tr}(\hat{W} \hat{A}_1).$$

You may disagree with the usage of the classical form of conditional probability, but whenever the “naked” quantum probabilities are testified in an experiment they are compared with the relative frequencies of the outcomes *relative to* the performance of the measurement. Similarly, for the conjunctions of outcomes we have

$$p\left(\bigwedge_{i \in I} A_i\right) = p\left(\bigwedge_{i \in I} a_i\right) \cdot \text{tr}\left(\hat{W} \prod_{i \in I} \hat{A}_i\right).$$

For any $I_1, I_2 \subset 2^{\{1, \dots, n\}} \setminus \emptyset$:

$$\begin{aligned} p\left(\left(\bigwedge_{i \in I_1} A_i\right) \wedge \left(\bigwedge_{j \in I_2} a_j\right)\right) &= p\left(\left(\bigwedge_{i \in I_1} A_i\right) \wedge \left(\bigwedge_{j \in I_1 \cup I_2} a_j\right)\right) \\ &= p\left(\bigwedge_{i \in I_1} A_i \mid \bigwedge_{j \in I_1 \cup I_2} a_j\right) \cdot p\left(\bigwedge_{j \in I_1 \cup I_2} a_j\right) \\ &= p\left(\bigwedge_{j \in I_1 \cup I_2} a_j\right) \cdot \text{tr}\left(\hat{W} \prod_{i \in I_1} \hat{A}_i\right). \end{aligned}$$

We used, for the first equality, the fact that it is impossible to observe the event A_i without performing the experiment a_i . Notice, that these expressions are valid even if the measurements are not compatible, then both sides are zero.

Thus we are ready to prove the main theorem.

3.5 Proof of the main theorem.

We claim that the generalized correlation vector \mathbf{p} that contains the probabilities of events $A_1, A_2, \dots, A_n, a_1, a_2, \dots, a_n$ and all their conjunctions is Kolmogorovian.

Theorem 3.1 *If $\tilde{\mathbf{p}}$, the probability distribution of the performances of the measurements is classical, has the decomposition (7), and if from the incompatibility of two measurements i, j (i.e. $[\hat{A}_i, \hat{A}_j] \neq 0$) follows that they are not performed together then the effective probabilities associated to the events $A_1, A_2, \dots, A_n, a_1, a_2, \dots, a_n$ and all their conjunctions admit a Kolmogorovian representation.*

Proof: In order to prove the theorem, we shall build an explicit Kolmogorovian representation for the effective correlation vector. Let us introduce the disjoint union of Ω^J -s defined above:

$$\Omega = \bigcup_{J \in \mathcal{K}} \Omega^J,$$

and let Σ be the σ -algebra on Ω generated by $\bigcup_{J \in \mathcal{K}} \Sigma^J$. We can extend the measures μ^J onto Σ in a natural way:

$$\mu^J(X) = \mu^J(X \cap \Omega^J), \quad (X \in \Sigma)$$

and define a new probability measure as

$$\mu = \sum_{J \in \mathcal{K}} \kappa_{\varepsilon^J} \cdot \mu^J.$$

It is easy to check that this is a Kolmogorovian measure. The representative sets of A_i -s and a_j -s are constructed as

$$X_{A_i} = \bigcup_{J \in \mathcal{K}: i \in J} X_{A_i}^J, \quad X_{a_j} = \bigcup_{J \in \mathcal{K}: j \in J} \Omega^J$$

According to the definition, $X_{A_i} \subset X_{a_j}$ for every $i \leq n$, and if for some i and j the respective measurements are not compatible then $X_{A_i} \cap X_{a_j} = \emptyset$ because i and j cannot be in the same J of \mathcal{K} . So μ gives 0 probability for them. For arbitrary $I_1, I_2 \in 2^{\{1, \dots, n\}} \setminus \{\emptyset\}$:

$$\begin{aligned} \bigcap_{i \in I_1} X_{A_i} &= \bigcap_{i \in I_1} \bigcup_{J_1 \in \mathcal{K}: i \in J_1} X_{A_i}^{J_1} = \\ \left\{ x \mid (\forall i \in I_1)(\exists J_1 \in \mathcal{K}) : i \in J_1 \text{ and } x \in X_{A_i}^{J_1} \right\} &= \\ \left\{ x \mid \exists J_1 \in \mathcal{K} : I_1 \subset J_1 \text{ and } x \in \bigcap_{i \in I_1} X_{A_i}^{J_1} \right\} &= \\ \bigcup_{J_1 \in \mathcal{K}: I_1 \subset J_1} \left(\bigcap_{i \in I_1} X_{A_i}^{J_1} \right) \end{aligned}$$

and

$$\bigcap_{j \in I_2} X_{a_j} = \bigcap_{j \in I_2} \bigcup_{J_2 \in \mathcal{K}: j \in J_2} \Omega^{J_2} =$$

$$\begin{aligned}
\left\{x \mid (\forall j \in I_2)(\exists J_2 \in \mathcal{K}) : j \in J_2 \text{ and } x \in \Omega^{J_2}\right\} = \\
\left\{x \mid \exists J_2 \in \mathcal{K} : I_2 \subset J_2 \text{ and } x \in \Omega^{J_2}\right\} = \\
\bigcup_{J_2 \in \mathcal{K}: I_2 \subset J_2} \Omega^{J_2}
\end{aligned}$$

This means that

$$\left(\bigcap_{i \in I_1} X_{A_i}\right) \cap \left(\bigcap_{j \in I_2} X_{a_j}\right) = \bigcup_{J \in \mathcal{K}: I_1 \cup I_2 \subset J} \left(\bigcap_{i \in I_1} X_{A_i}^J\right). \quad (10)$$

If $I_1 \cup I_2$ contains incompatible measurements then (10) gives \emptyset , because there is no $J \in \mathcal{K}$ containing $I_1 \cup I_2$. But if $I_1 \cup I_2 \in \mathcal{K}$ then

$$\begin{aligned}
& \mu \left(\left(\bigcap_{i \in I_1} X_{A_i} \right) \cap \left(\bigcap_{j \in I_2} X_{a_j} \right) \right) = \\
& \mu \left(\bigcup_{J \in \mathcal{K}: I_1 \cup I_2 \subset J} \left(\bigcap_{i \in I_1} X_{A_i}^J \right) \right) = \\
& \sum_{J \in \mathcal{K}: I_1 \cup I_2 \subset J} \kappa_{\varepsilon^J} \cdot \mu^J \left(\bigcap_{i \in I_1} X_{A_i}^J \right) = \\
& \left(\sum_{J \in \mathcal{K}: I_1 \cup I_2 \subset J} \kappa_{\varepsilon^J} \right) \cdot \text{tr} \left(\hat{W} \prod_{i \in I_1} \hat{A}_i \right) = \\
& p \left(\bigwedge_{j \in I_1 \cup I_2} a_j \right) \cdot \text{tr} \left(\hat{W} \prod_{i \in I_1} \hat{A}_i \right).
\end{aligned}$$

which is exactly the same as the effective probability of the event $(\bigwedge_{i \in I_1} A_i) \wedge (\bigwedge_{j \in I_2} a_j)$ obtained at the beginning of this section.

We get similarly that for some $I \in \mathcal{K}$:

$$\mu \left(\bigcap_{i \in I} X_{a_i} \right) = p \left(\bigwedge_{i \in I} a_i \right)$$

and

$$\mu \left(\bigcap_{i \in I} X_{A_i} \right) = p \left(\bigwedge_{i \in I} a_i \right) \cdot \text{tr} \left(\hat{W} \prod_{i \in I} \hat{A}_i \right),$$

showing the theorem.

□

This last form shows that if we denote the correlation vector $I \mapsto \text{tr}(\hat{W} \prod_{i \in I} \hat{A}_i)$ with the symbol π , then we have a Kolmogorovian representation for the vector $\tilde{\mathbf{p}} \cdot \pi$, in accordance with the notations of the Kolmogorovian censorship hypothesis.

For example, the Kolmogorovian representation associated to the Bell-like experiment described at the beginning of this work is given in Table 2.

$A \cap B$ $\frac{3}{32}$	$A \cap \neg B$ $\frac{1}{32}$	$A \cap B'$ $\frac{3}{32}$	$A \cap \neg B'$ $\frac{1}{32}$
$\neg A \cap B$ $\frac{1}{32}$	$\neg A \cap \neg B$ $\frac{3}{32}$	$\neg A \cap B'$ $\frac{1}{32}$	$\neg A \cap \neg B'$ $\frac{3}{32}$
$A' \cap B$ 0	$A' \cap \neg B$ $\frac{1}{8}$	$A' \cap B'$ $\frac{3}{32}$	$A' \cap \neg B'$ $\frac{1}{32}$
$\neg A' \cap B$ $\frac{1}{8}$	$\neg A' \cap \neg B$ 0	$\neg A' \cap B'$ $\frac{1}{32}$	$\neg A' \cap \neg B'$ $\frac{3}{32}$

Table 2: The Kolmogorovian representation of the Orsay effective frequencies.

4 Conclusion.

We shall not, in our conclusion, discuss in details the possible interpretations of our result. Some of them can be found in other articles. We just mention that this representation can be used for the proof of the existence of a local deterministic hidden variable model (Durt 1995, 1996a,b, Szabó 1995 a,b).

The aim of the article was to prove Szabó's Kolmogorovian Censorship hypothesis. We were able to do this by taking into account only observable events, that is, the performances of the measurements and the beeps of the detectors. We supposed that the incompatible measurements are not carried out together and that the probability distribution of the performances of the measurements is classical. This way the quantum probabilities appear as classical conditional probabilities. It is useful to remark that because the events of carrying out the measurements are taken into the event algebra, in hidden variable models the choice of an experiment (the choice of the

direction of the magnet in our example) is dependent on the value taken by the hidden variable. This has important consequences for the question of determinism and free-will (Durt 1995, 1996a,b; Szabó 1995a,b).

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